In a nutshell: Approximating solutions to systems of higher-order initial value problems

Given a system of m initial-value problems (IVP) where the k^{th} IVP is an n_k-order IVP, then such a system can be converted into system of $n_1 + \ldots + n_m$ 1^{st}-order IVPs by generalizing the steps seen previously. This cannot be described mathematically without a lot of unnecessary notation, and thus, instead, we will give an example. Here is a system of two 2^{nd}-order IVPs and a 3^{rd}-order IVP.

$$
\begin{aligned}
x^{(2)}(t) &= f_1\left(t, x(t), x^{(1)}(t), y(t), y^{(1)}(t), z(t), z^{(1)}(t), z^{(2)}(t)\right)\\
y^{(2)}(t) &= f_2\left(t, x(t), x^{(1)}(t), y(t), y^{(1)}(t), z(t), z^{(1)}(t), z^{(2)}(t)\right)\\
z^{(3)}(t) &= f_3\left(t, x(t), x^{(1)}(t), y(t), y^{(1)}(t), z(t), z^{(1)}(t), z^{(2)}(t)\right)\\
x(t_0) &= x_0\\
x^{(1)}(t_0) &= x_0^{(1)}\\
y(t_0) &= y_0\\
y^{(1)}(t_0) &= y_0^{(1)}\\
z(t_0) &= z_0\\
z^{(1)}(t_0) &= z_0^{(1)}\\
z^{(2)}(t_0) &= z_0^{(2)}
\end{aligned}
$$

This can be written as the following system of seven IVPs:

$$
\mathbf{w}(t) = \begin{pmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\\ w_5(t)\\ w_6(t)\\ w_7(t) \end{pmatrix} = \begin{pmatrix} x(t)\\ x^{(1)}(t)\\ y(t)\\ y^{(1)}(t)\\ z(t)\\ z^{(1)}(t)\\ z^{(2)}(t) \end{pmatrix},\ \mathbf{w}^{(1)}(t) = \mathbf{f}\left(t, \mathbf{w}(t)\right) = \begin{pmatrix} w_2(t)\\ f_1\left(t, w_1(t), \ldots, w_7(t)\right)\\ w_4(t)\\ f_2\left(t, w_1(t), \ldots, w_7(t)\right)\\ w_6(t)\\ w_7(t)\\ f_3\left(t, w_1(t), \ldots, w_7(t)\right) \end{pmatrix}, \text{ and } \mathbf{w}(t_0) = \begin{pmatrix} x_0\\ x_0^{(1)}\\ y_0\\ y_0^{(1)}\\ z_0\\ z_0^{(1)}\\ z_0^{(2)} \end{pmatrix} = \mathbf{w}_0 .
$$

where we index **w** from 1 to 7 = 2 + 2 + 3. When we examine the approximation $\mathbf{w}_k$, the 1^{st} entry approximates $x(t_k)$, the 3^{rd} entry approximates $y(t_k)$, and the 5^{th} entry approximates $z(t_k)$. The approximations of the 2^{nd}, 4^{th}, and 6^{th} entries can be used for the derivatives with respect to the splines used to find intermediate values.